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FLOW OF A VISCOELASTIC FLUID BETWEEN TWO ROTATING CIRCULAR CYLINDERS SUBJECT TO SUCTION OR INJECTION

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SUMMARY

The steady flow of an Oldroyd-B fluid between two porous concentric circular cylinders is studied. The equation of motion and the constitutive equations form a system of non-linear ODEs that is solved numerically, and in a few cases the numerical results are compared with a known analytical solution. We consider the effect of the non-Newtonian nature of the fluid on the drag and on the boundary layer structure near the walls. Numerical computations show the effect of the non-Newtonian quantities on the velocity and on the shear stress as the dimensionless parameters are varied.

KEY WORDS: Oldroyd–B; non-Newtonian; boundary layer; collocation; spline

1. INTRODUCTION

Laminar flow of a linearly viscous fluid between two coaxial rotating cylinders of infinite length takes place along circular streamlines (cylindrical Couette flow) if a non-dimensional number associated with the rotation speed of the cylinders does not exceed a critical value, known as the Taylor number,[1–3] and an analytical solution is available in all texts on fluid mechanics (see, e.g. References 4 and 5).

The inadequacy of the Navier–Stokes theory in describing rheologically complex fluids used in industrial processing, such as polymer solutions, melts and paints, has led to the formulation of other mathematical models able to predict the flow of such materials. One of them is the Oldroyd-B fluid model.[6,7] This fluid, which takes into account elastic and memory effects exhibited by most polymeric and biological liquids, has been used quite widely in many applications and the results of simulations fit experimental data in a wide range.[8]

Furthermore, it is known that even for the Navier–Stokes fluid, if the cylinder surfaces are porous, a uniform suction applied on it can sensibly change the boundary layer structure, reduce the drag and hinder viscous diffusion of vorticity.[5,9,10] The flows of many other fluid models have been studied in this geometry, but we shall not discuss them here.[11–15]

Here we shall consider the flow of an Oldroyd-B fluid between two concentric circular cylinders subject to suction or injection at the inner porous wall. As a particular case we also consider the case

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of a porous cylinder rotating in a fluid of infinite extent subject to suction at the wall. In the case of the Navier–Stokes fluid, this problem was first studied by Hamel,[16] and later analyzed by Preston[17] and Thwaites.[18]

Our interest is in understanding the interaction between the viscous and elastic mechanisms and the effect of suction or injection at the boundary. Results are presented for many cases and compared with those that are available in the literature. Only in a few cases is it possible to solve the equations analytically; in all other cases the use of a numerical method is unavoidable.

The non-linear ODEs which couple the velocity with the stress are cast with appropriate boundary conditions and solved by a collocation method using spline approximation functions. We recover the classical Hamel result, for the Navier–Stokes fluid, by letting the outer radius tend to infinity.

2. FORMULATION OF THE PROBLEM

The stress tensor in an Oldroyd-B fluid is given by

$$\mathbf{T} = -p\mathbf{I} + \mathbf{S}, \qquad \mathbf{S} + \lambda_1(\dot{\mathbf{S}} - \mathbf{LS} - \mathbf{SL}^{\mathrm{T}}) = \mu[\mathbf{A}_1 + \lambda_2(\dot{\mathbf{A}}_1 - \mathbf{LA}_1 - \mathbf{A}_1\mathbf{L}^{\mathrm{T}})], \tag{1}$$

where $-p\mathbf{I}$ is the spherical part of the stress due to the constraint of incompressibility, the dot denotes a material time derivative, μ is the viscosity and λ_1 and λ_2 are material time constants referred to respectively as the relaxation and retardation times. It is assumed that $\lambda_1 \geqslant \lambda_2 \geqslant 0$.

The tensors $\mathbf{L}$ and $\mathbf{A}_1$ are defined as

$$\mathbf{L} = \mathrm{grad}\mathbf{v}, \qquad \mathbf{A}_1 = \mathbf{L} + \mathbf{L}^{\mathrm{T}}.$$

It should be noted that this model includes the classical linearly viscous Navier–Stokes fluid as a special case for $\lambda_1 = \lambda_2 = 0$, and to the Maxwell fluid when $\lambda_2 = 0$.

If the fluid is assumed to be incompressible, then

$$\mathrm{div}\ \mathbf{v} = 0 \tag{2}$$

holds.

Let us now consider the motion of an Oldroyd-B fluid between two coaxial, circular, infinite cylinders rotating steadily around their common axis. We shall denote by R_1 and R_2 the radii of the inner and outer cylinders respectively and by Ω_1 and Ω_2 their angular velocities (the subscripts 1 and 2 will indicate quantities on the inner and outer boundaries respectively). Let (r, θ, z) be a cylindrical co-ordinate system with the z-axis coincident with the axis of the cylinders and $\theta > 0$ anticlockwise (see Figure 1). Let us assume $\Omega_1 > 0$.

We shall seek an axisymmetric two-dimensional solution and thus assume that all variables depend on r only.

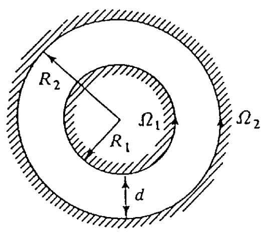

Figure 1. Geometry of problem: cross-section of cylinders

Let us indicate the stress tensor and the velocity as

$$\mathbf{S} = \begin{pmatrix} S_{rr} & S_{r\theta} \\ S_{r\theta} & S_{\theta\theta} \end{pmatrix}, \qquad \mathbf{v} = \begin{pmatrix} u \\ v \end{pmatrix}.$$

The two scalar momentum equations for steady flows without body forces are

$$\rho\left(u\frac{\mathrm{d}u}{\mathrm{d}r} - \frac{v^2}{r}\right) = -\frac{\mathrm{d}p}{\mathrm{d}r} + \frac{\mathrm{d}S_{rr}}{\mathrm{d}r} + \frac{S_{rr} - S_{\theta\theta}}{r}, \tag{3}$$

$$\rho\left(u\frac{\mathrm{d}v}{\mathrm{d}r} + \frac{uv}{r}\right) = \frac{\mathrm{d}S_{r\theta}}{\mathrm{d}r} + \frac{2S_{r\theta}}{r}, \tag{4}$$

where ρ is the mass density and $R_1 \leqslant r \leqslant R_2$.

Also, (1) reduces to

$$S_{rr} + \lambda_1\left(u\frac{\mathrm{d}S_{rr}}{\mathrm{d}r} - 2\frac{\mathrm{d}u}{\mathrm{d}r}S_{rr}\right) = 2\mu\left\{\frac{\mathrm{d}u}{\mathrm{d}r} + \lambda_2\left[u\frac{u^2u}{\mathrm{d}r^2} - 2\left(\frac{\mathrm{d}u}{\mathrm{d}r}\right)^2\right]\right\}, \tag{5}$$

$$S_{r\theta} + \lambda_1\left[u\frac{\mathrm{d}S_{r\theta}}{\mathrm{d}r} - \left(\frac{\mathrm{d}v}{\mathrm{d}r} - \frac{v}{r}\right)S_{rr} - \left(\frac{\mathrm{d}u}{\mathrm{d}r} + \frac{u}{r}\right)S_{r\theta}\right] = \mu\left\{\frac{\mathrm{d}v}{\mathrm{d}r} - \frac{v}{r} + \lambda_2\left[u\frac{\mathrm{d}^2v}{\mathrm{d}r^2} - \left(3\frac{\mathrm{d}u}{\mathrm{d}r} + 2\frac{u}{r}\right)\left(\frac{\mathrm{d}v}{\mathrm{d}r} - \frac{v}{r}\right)\right]\right\}, \tag{6}$$

$$S_{\theta\theta} + \lambda_1\left[u\frac{\mathrm{d}S_{\theta\theta}}{\mathrm{d}r} - 2\frac{u}{r}S_{\theta\theta} - 2\left(\frac{\mathrm{d}v}{\mathrm{d}r} - \frac{v}{r}\right)S_{r\theta}\right] = 2\mu\left\{\frac{u}{r} + \lambda_2\left[\frac{u}{r}\frac{\mathrm{d}u}{\mathrm{d}r} - 3\frac{u^2}{r^2} - \left(\frac{\mathrm{d}v}{\mathrm{d}r} - \frac{v}{r}\right)^2\right]\right\}. \tag{7}$$

The two boundary conditions

$$v(R_1) = R_1\Omega_1, \qquad v(R_2) = R_2\Omega_2 \tag{8}$$

corresponding to the 'no-slip' condition are prescribed.

Let us assume that the cylinders have porous walls and a constant velocity u_1 is applied to the inner one, denoting a uniform suction ($u_1 < 0$) or injection ($u_1 > 0$). Generally, $|u_1|$ is small compared with $R_1\Omega_1$.

From the constraint of incompressibility (2), we have

$$\frac{\mathrm{d}(ru)}{\mathrm{d}r} = 0,$$

i.e.

$$u = \frac{R_1 u_1}{r}. \tag{9}$$

Remark 1

It follows that $u_2 = R_1u_1/R_2$, i.e. the outer wall is also subject to a radial velocity. A suction at the inner wall corresponds to an injection at the outer one and vice versa.

In particular, if $u_1 = 0$, then $u_2 = 0$ and $u \equiv 0$ everywhere.

3. NON-DIMENSIONAL EQUATIONS

Let us non-dimensionalize the above equations by introducing the change of variables

$$r \to \frac{r}{R_1}, \qquad u \to \frac{u}{u_1} \quad (\text{if } u_1 \neq 0), \qquad v \to \frac{v}{R_1\Omega_1},$$
$$S_{ij} \to \frac{S_{ij}}{\rho(R_1\Omega_1)^2}, \qquad p \to \frac{p}{\rho(R_1\Omega_1)^2} \tag{10}$$

and defining the dimensionless constants

$$Re = \frac{\Omega_1 R_1^2}{\nu}, \qquad \alpha = \frac{u_1}{R_1\Omega_1}, \qquad \beta = \frac{R_2}{R_1}, \qquad \gamma = \frac{\Omega_2}{\Omega_1},$$
$$c_1 = \lambda_1\Omega_1, \qquad c_2 = \lambda_2\Omega_1, \qquad k_1 = \frac{u_1\lambda_1}{R_1}, \qquad k_2 = \frac{u_1\lambda_2}{R_1},$$

with $\nu = \mu/\rho$ the kinematic viscosity. After substitution, equations (3)–(7) become

$$\alpha^2 u\frac{\mathrm{d}u}{\mathrm{d}r} - \frac{v^2}{r} = -\frac{\mathrm{d}p}{\mathrm{d}r} + \frac{\mathrm{d}S_{rr}}{\mathrm{d}r} + \frac{S_{rr} - S_{\theta\theta}}{r}, \tag{11}$$

$$\alpha\left(u\frac{\mathrm{d}v}{\mathrm{d}r} + \frac{uv}{r}\right) = \frac{\mathrm{d}S_{r\theta}}{\mathrm{d}r} + \frac{2S_{r\theta}}{r}, \tag{12}$$

$$S_{rr} + k_1\left(u\frac{\mathrm{d}S_{rr}}{\mathrm{d}r} - 2\frac{\mathrm{d}u}{\mathrm{d}r}S_{rr}\right) = \frac{2\alpha}{Re}\left\{\frac{\mathrm{d}u}{\mathrm{d}r} + k_2\left[u\frac{\mathrm{d}^2u}{\mathrm{d}r^2} - 2\left(\frac{\mathrm{d}u}{\mathrm{d}r}\right)^2\right]\right\}, \tag{13}$$

$$S_{r\theta} + k_1\left[u\frac{\mathrm{d}S_{r\theta}}{\mathrm{d}r} - \left(\frac{u}{r} + \frac{\mathrm{d}u}{\mathrm{d}r}\right)S_{r\theta}\right] - c_1\left(\frac{\mathrm{d}v}{\mathrm{d}r} - \frac{v}{r}\right)S_{rr}$$
$$= \frac{1}{Re}\left(\frac{\mathrm{d}v}{\mathrm{d}r} - \frac{v}{r}\right) + \frac{k_2}{Re}\left[u\frac{\mathrm{d}^2v}{\mathrm{d}r^2} - \left(3\frac{\mathrm{d}u}{\mathrm{d}r} + 2\frac{u}{r}\right)\left(\frac{\mathrm{d}v}{\mathrm{d}r} - \frac{v}{r}\right)\right], \tag{14}$$

$$S_{\theta\theta} + k_1\left(u\frac{\mathrm{d}S_{\theta\theta}}{\mathrm{d}r} - 2\frac{u}{r}S_{\theta\theta}\right) - 2c_1\left(\frac{\mathrm{d}v}{\mathrm{d}r} - \frac{v}{r}\right)S_{r\theta} = \frac{2\alpha}{Re}\left[\frac{u}{r} + k_2\left(\frac{u}{r}\frac{\mathrm{d}u}{\mathrm{d}r} - 3\frac{u^2}{r^2}\right)\right] - \frac{2c_2}{Re}\left(\frac{\mathrm{d}v}{\mathrm{d}r} - \frac{v}{r}\right)^2 \tag{15}$$

to be solved for $1 \leqslant r \leqslant \beta$. Equation (11) is used to get the pressure gradient $\mathrm{d}p/\mathrm{d}r$. From (9) and (10) it follows that for $u_1 \neq 0$

$$u = \frac{1}{r}. \tag{16}$$

By substituting (16) into (12)–(15), we finally get the four differential equations

$$\alpha\left(\frac{\mathrm{d}v}{\mathrm{d}r}+\frac{v}{r}\right)=r\frac{\mathrm{d}S_{r\theta}}{\mathrm{d}r}+2S_{r\theta}, \tag{17}$$

$$\left(1+\frac{2k_1}{r^2}\right)S_{rr}+\frac{k_1}{r}\frac{\mathrm{d}S_{rr}}{\mathrm{d}r}=\frac{-2\alpha}{Rer^2}, \tag{18}$$

$$S_{r\theta}+\frac{k_1}{r}\frac{\mathrm{d}S_{r\theta}}{\mathrm{d}r}-c_1\left(\frac{\mathrm{d}v}{\mathrm{d}r}-\frac{v}{r}\right)S_{rr}=\frac{1}{Re}\left[\frac{k_2}{r}\frac{\mathrm{d}^2v}{\mathrm{d}r^2}+\left(1+\frac{k_2}{r^2}\right)\left(\frac{\mathrm{d}v}{\mathrm{d}r}-\frac{v}{r}\right)\right], \tag{19}$$

$$\left(1-\frac{2k_1}{r^2}\right)S_{\theta\theta}+\frac{k_1}{r}\frac{\mathrm{d}S_{\theta\theta}}{\mathrm{d}r}-2c_1\left(\frac{\mathrm{d}v}{\mathrm{d}r}-\frac{v}{r}\right)S_{r\theta}=\frac{2}{Re}\left[\frac{\alpha}{r^2}-\frac{4k_2\alpha}{r^4}-c_2\left(\frac{\mathrm{d}v}{\mathrm{d}r}-\frac{v}{r}\right)^2\right] \tag{20}$$

with the boundary conditions

$$v(1)=1, \qquad v(\beta)=\beta\gamma. \tag{21}$$

The boundary conditions for stresses are obtained by evaluating (18)–(20) on the boundary, assuming that this can be done.

The linear equation (18) is unrelated to the others and admits the family of solutions

$$S_{rr}=G\frac{\exp(-r^2/2k_1)}{r^2}-\frac{2\alpha}{Re\;r^2}.$$

Since the solution is continuous in k_1, it follows that $G\equiv 0$ and the solution becomes

$$S_{rr}=\frac{-2\alpha}{Re\;r^2}, \tag{22}$$

which turns out to be independent of k_1. In fact, S_{rr} is identical for all fluids having the same viscosity.

By substituing (22) into (19),we get a system of three ODEs (17), (19) and (20) which combines in a non-linear way the three variables v, $S_{r\theta}$ and $S_{\theta\theta}$.

The classical Couette flow (Newtonian fluid without suction) is obtained as a special case.

The next section is devoted to other simpler cases, for which the equations will be reduced and an exact solution can easily be found. In Section 5 the general case will be analysed.

4. SPECIAL CASES

In this section three special cases of the problem are considered.

Case A. Oldroyd-B fluid without suction or injection

It is easy to prove that v satisfies the second-order BVP

$$\frac{\mathrm{d}^2v}{\mathrm{d}r^2}+\frac{1}{r}\frac{\mathrm{d}v}{\mathrm{d}r}-\frac{v}{r^2}=0 \tag{23}$$

with (21). It admits the unique solution[19]

$$v(r) = \frac{\beta^2\gamma - 1}{\beta^2 - 1} r + \frac{\beta^2(1-\gamma)}{\beta^2 - 1}\frac{1}{r}, \tag{24}$$

$$S_{r\theta} = \frac{2\beta^2(\gamma - 1)}{Re(\beta^2 - 1)}\frac{1}{r^2}, \qquad S_{\theta\theta} = \frac{2(c_1 - c_2)}{Re}\frac{4\beta^4(\gamma-1)^2}{(\beta^2-1)^2 r^4} \geqslant 0 \tag{25}$$

It is worth noting that the velocity field and shear stress depend only on geometrical and physical parameters, but not on the relaxation and retardation times, and match those in the Newtonian case.[4] This happens also for the plane Couette flow.

Only the normal stress $S_{\theta\theta}$ depends on the non-Newtonian fluid, being zero in the Navier–Stokes case.

Remark 2

It is easy to prove that $v(r) \leqslant \max(v(1), v(\beta))$.*

The dimensionless torque per unit height exerted on the fluid inside the cylinder $r = \text{const.}$ is given by

$$T = 2\pi r^2 S_{r\theta} = \frac{4\pi\beta^2(\gamma - 1)}{Re(\beta^2 - 1)}, \tag{26}$$

which is a constant ($\lesseqgtr 0$ iff $\gamma \lesseqgtr 1$) independent of r.†

Case B. Newtonian fluid with suction or injection

After straightforward computations it follows that v satisfies the BVP

$$\frac{\mathrm{d}^2 v}{\mathrm{d}r^2} + \frac{1-q}{r}\frac{\mathrm{d}v}{\mathrm{d}r} - \frac{1+q}{r^2} v = 0 \tag{27}$$

with boundary conditions (21), where

$$q = \alpha Re = \frac{R_1 u_1}{\nu}.$$

It easy to verify that it admits the unique solution:[19]

$$v(r) = \frac{\beta^2\gamma - 1}{\beta^{q+2} - 1} r^{q+1} + \frac{\beta^2(\beta^q - \gamma)}{\beta^{q+2} - 1}\frac{1}{r} \quad \text{for } q \neq -2, \tag{28}$$

$$v^*(r) = \frac{\beta^2\gamma - 1}{\log\beta}\frac{\log r}{r} + \frac{1}{r} \quad \text{for } q = -2 \tag{29}$$

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*In particular we have

$$\text{for } \gamma \geqslant \frac{1}{\beta}\begin{cases} v(r) \leqslant v(\beta), & \\ v(r) \geqslant v(1) & \text{if } \gamma > (\beta^2+1)/2\beta^2 \end{cases},$$

$$\text{for } \gamma \leqslant \frac{1}{\beta}\begin{cases} v(r) \geqslant v(\beta), & \text{if } \gamma < 1/\beta^2 \\ v(r) \leqslant v(1) & \end{cases}.$$

Moreover, we can directly prove that $v(r) \geqslant 0$ if $\gamma \geqslant 0$.

†Formula (26) is often used experimentally for the determination of the kinematic viscosity.[8]

and that

$$\lim_{q\to -2} v(r) = v^*(r) \quad \text{and} \quad v(r) \geqslant 0 \quad \text{if } \gamma \geqslant 0 \quad \text{for every } q.$$

The dimensionless shear stress is

$$S_{r\theta} = \begin{cases} \dfrac{1}{Re}\left(\dfrac{\beta^2\gamma - 1}{\beta^{q+2} - 1} q r^q - 2\dfrac{\beta^2(\beta^q - \gamma)}{\beta^{q+2} - 1}\dfrac{1}{r^2}\right) & \text{for } q \neq -2, \\ \dfrac{1}{Re}\left(\dfrac{\beta^2\gamma - 1}{\log\beta}\dfrac{1 - 2\log r}{r^2} - \dfrac{2}{r^2}\right) & \text{for } q = -2, \end{cases} \tag{30}$$

$$S_{\theta\theta} = \frac{2\alpha}{Re\, r^2}.$$

In the case of one porous cylinder immersed in an infinite fluid, this solution matches that obtained by Hamel[16] (see also Case C).

Unlike Case A, the torque $T = 2\pi r^2 S_{r\theta}$ is not uniform on the whole domain. However, with a suitable choice of the parameters (β, γ, q) it can vanish at some points. Since (30) is linear in γ, it follows that at the two extrema

$$S_{r\theta}|_{r=1} = 0 \quad \text{for } \gamma = \begin{cases} \dfrac{q + 2\beta^{q+2}}{\beta^2(q+2)}, & q \neq -2, \\ \dfrac{1 + 2\log\beta}{\beta^2}, & q = -2, \end{cases}$$

$$S_{r\theta}|_{r=\beta} = 0 \quad \text{for } \gamma = \begin{cases} \dfrac{\beta^q(q+2)}{\beta^{q+2} q + 2}, & q \neq -2, \\ \dfrac{1}{\beta^2(1 - 2\log\beta)}, & q = -2. \end{cases}$$

More generally, by varying the angular velocity of the outer cylinder, we can tune S_{rr} on the inner or the outer cylinder to some specified value. Unlike Case A, here the solution v depends on the fluid viscosity through the parameter q; consequently, it follows that a reduction of $|u_1|$ has the same effect as an increase in v (see also Section 6). This means that the boundary layer structure for a fluid of a given viscosity can be controlled by imposing a local suction.

The normal stresses S_{rr} and $S_{\theta\theta}$ are opposite and independent of the velocity field. The suction (or injection) velocity produces a normal stress difference $S_{rr} - S_{\theta\theta} < 0$, as experiments confirm.

Note that, if $u_1 = 0$ ($\Rightarrow \alpha = q = 0$), (27), (28) and (30) reduce to (23)–(25) respectively and $S_{rr} = S_{\theta\theta} = 0$.

In order to better understand the effect of suction and injection at $r = 1$, let us study the dependence on q of the function

$$G(q) = \left.\frac{\mathrm{d}v}{\mathrm{d}r}\right|_{r=1} = \begin{cases} \dfrac{\beta^2\gamma - 1}{\beta^{q+2} - 1}(q+2) - 1, & q \neq -2, \\ \dfrac{\beta^2\gamma - 1}{\log\beta} - 1, & q = -2. \end{cases}$$

We have that

(i) $G(q) \lesseqgtr -1$ iff $\beta^2\gamma - 1 \gtreqless 0$
(ii) $G(q) \underset{q \gg 0}{\longrightarrow} -1$
(iii) $G(q)/q \underset{q \ll -2}{\longrightarrow} 1 - \beta^2\gamma$

meaning that $|G(q)|$ increases in an almost linear manner with the slope $|1 - \beta^2\gamma|$.

An increase in the q-values, ranging from negative to positive, causes a lesser slope for $|v|$ and therefore a thickening of the boundary layer (see Figure 2). However, there exists a critical value of $q > 0$ beyond which there is no noticeable reduction of G. It follows that the effectiveness of a radial velocity is greater when $u_1 < 0$ (suction). The only exception is when $\gamma = 1/\beta^2$: in this case we have $G(q) = -1$ and neither suction nor injection affects the solution.

Case C. One cylinder rotating in an infinite fluid

In another case an analytical solution of the system (17)–(21) can be found. This happens if

(i) $\gamma = 1/\beta^2$ and particularly when
(ii) $\beta \to \infty$, $\gamma = 0$, $v(\beta) = \beta\gamma = 0$.

The latter case concerns the important flow given by a porous cylinder rotating in an infinite fluid. In both cases one easily verify that the unique solution for v and $S_{r\theta}$ is:

$$v(r) = \frac{1}{r}, \qquad S_{r\theta} = \frac{-2}{Re\, r^2}$$

and $S_{\theta\theta}$ is the solution of the ODE

$$\left(1 - \frac{2k_1}{r^2}\right) S_{\theta\theta} + \frac{k_1}{r}\frac{\mathrm{d}S_{\theta\theta}}{\mathrm{d}r} = \frac{2}{Re}\left(\frac{\alpha}{r^2} + 4\frac{c_1 - c_2 - k_2\alpha}{r^4}\right).$$

Therefore it turns out that v and $S_{r\theta}$ depend neither on the normal velocity nor on the fluid nature and match the classical solution in References 4 and 16. This solution can also be recovered in Cases A and B.

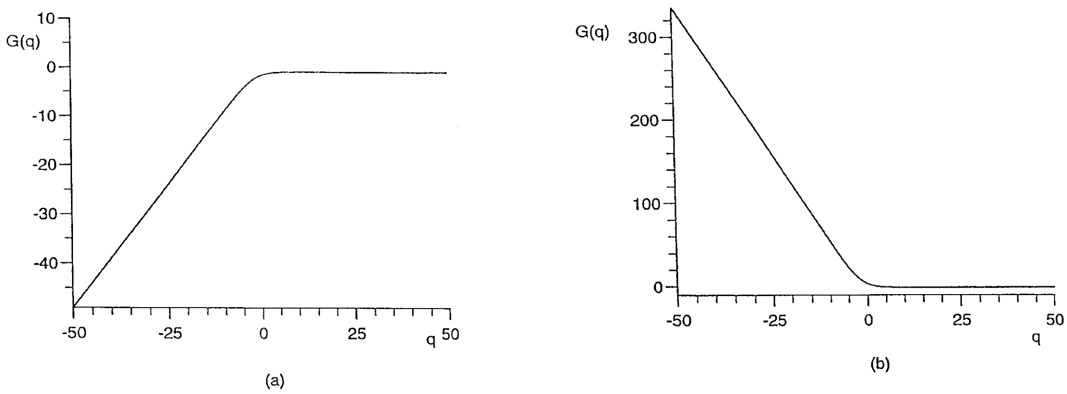

Figure 2. Plots of $G(q)$ for (a) $\beta = 2$, $\gamma = 0$ and (b) $\beta = 2$, $\gamma = 2$

5. NUMERICAL METHOD

In the general case the three coupled ODEs (17), (19) and (20), rewritten as

$$\alpha\left(\frac{\mathrm{d}v}{\mathrm{d}r}+\frac{v}{r}\right)=r\frac{\mathrm{d}S_{r\theta}}{\mathrm{d}r}+2S_{r\theta}, \tag{31}$$

$$S_{r\theta}+\frac{k_1}{r}\frac{\mathrm{d}S_{r\theta}}{\mathrm{d}r}=\frac{1}{Re}\left[\frac{k_2}{r}\frac{\mathrm{d}^2v}{\mathrm{d}r^2}+\left(1+\frac{k_2-2k_1}{r^2}\right)\left(\frac{\mathrm{d}v}{\mathrm{d}r}-\frac{v}{r}\right)\right], \tag{32}$$

$$\left(1-\frac{2k_1}{r^2}\right)S_{\theta\theta}+\frac{k_1}{r}\frac{\mathrm{d}S_{\theta\theta}}{\mathrm{d}r}-2c_1\left(\frac{\mathrm{d}v}{\mathrm{d}r}-\frac{v}{r}\right)S_{r\theta}=\frac{2}{Re}\left[\frac{\alpha}{r^2}-\frac{4k_2\alpha}{r^4}-c_2\left(\frac{\mathrm{d}v}{\mathrm{d}r}-\frac{v}{r}\right)^2\right], \tag{33}$$

are solved together in $(0, \beta)$ by a collocation method with spline approximation functions. The basis functions have been chosen as B-splines, which are known to have good properties of regularity and well conditioning. In linear cases the related algebraic systems have a banded structure and can be solved efficiently.[20]

Let us give a short description of the numerical method. We fix a set of breakpoints $\Delta\equiv(r_i)_{i=1,\ldots,l+1}$ such that $0\equiv r_1<r_2<\cdots<r_l<r_{l+1}\equiv\beta$. Let $h,\ p\geqslant 1$ be two integers, the first one being arbitrary and the second one denoting the highest-order derivative ($p=2$ in our case), and $n=hl+p$.

Let $T\equiv(t_i)_{i=1,\ldots,n+h+p}$ be a non-decreasing sequence of points containing r_1 and r_{l+1} $h+p$ times and all other breakpoints h times. The set of B-splines $(B_i)_{i=1,\ldots,n}$ built on the sequence T is a basis for the space $S_{h+p,\Delta}\equiv P_{h+p,\Delta}\cap C^{p-1}(0,\beta)$, where $P_{h+p,\Delta}$ is the space of piecewise polynomials on Δ of order $h+p$ and $C^{p-1}(0,\beta)$ is the space of differentiable functions on $(0,\beta)$ of order $p-1$.

We seek approximation functions

$$\hat{v}(r)=\sum_{i=1}^{n}\alpha_iB_i(r),\qquad \hat{\mathbf{S}}_j(r)=\sum_{i=1}^{n}\beta_i^jB_i(r),\qquad \hat{\mathbf{S}}=(\hat{\mathbf{S}}_j)_{j=1,2}=\begin{pmatrix}\hat{S}_{r\theta}\\ \hat{S}_{\theta\theta}\end{pmatrix}$$

such that

$$\mathscr{T}(\sigma,\hat{v},\hat{v}',\hat{v}'',\hat{\mathbf{S}},\hat{\mathbf{S}}')=0,$$

where $\mathscr{T}$ is the quasi-linear operator indicating the system of discretized ODEs (31)–(33), with boundary conditions (21), and σ stands for one of the parameters α, Re, k_1, k_2, c_1, c_2 to be changed one at a time.

Let us now choose a sequence of h collocation points $(\rho_{\mathrm{i}})_{i=1,\ldots,h}$ as the zeros of the hth-degree Legendre polynomial between two successive breakpoints. The coefficients (α_i) and (β_i^j) are then computed as a solution of the non-linear algebraic system

$$\mathscr{T}(\sigma,\hat{v},\hat{v}',\hat{v}'',\hat{\mathbf{S}},\hat{\mathbf{S}}')(\rho_i)=0\quad\text{for every } i. \tag{34}$$

Since the collocation points are equidistributed in each interval, the global error is minimal and its order is $O(|\xi|^{h+p})$, where $|\xi|$ is the maximum distance between successive breakpoints. At the breakpoints the order of convergence may be even better, the error being $O(|\xi|^{2h})$.[21]
Much care has to be given to the choice of the breakpoints: indeed, their location and their number are strictly dependent on the function to be approximated. In general they are more condensed near the walls, where the functions are supposed to change more rapidly (see Reference 20 for details). On the other hand, a proper selection of the order of the splines allows one to approximate quite stiff functions.

Since (34) is a non-linear system, we solved it with a Newton-like technique, taking a null solution, or the analytical solution given in Sections 4 and 5 for some values of σ, as an initial guess. Then a locally parametrized continuation method is applied along one parameter σ at a time, where an already computed solution was used as an initial guess for closely following σ.[22]

6. RESULTS

First of all we tested our algorithm on the special cases described in Section 4. Excellent agreement with the analytical solutions is obtained. Then we allowed one parameter σ to change at a time in (34) to investigate the flow response and its sensitivity to variations in the physical, geometrical and rheological features.

We fixed the order of the splines equal to six for v and S over eight equidistributed breakpoints. Then, in changing the parameter σ, we automatically redistributed the knots according to the shape of the previously computed solution. Their number should possibly increase until the solution settles down.

We noticed a strong correlation among the many parameters upon which the flow depends, in that some effects visible for some values are not evident any longer for others. The solution itself exists only for some combinations of parameters and within the limit of laminar flows. However, it seems that the solution varies continuously along all parameters.

In all experiments the normal stress $S_{\theta\theta}$ can reach an order of magnitude larger than the shear stress and is the most sensitive to variations in the parameters. We always have that $S_{rr} - S_{\theta\theta} < 0$.

Among a variety of numerical experiments we report here only the most significant: we always fixed $R_1 = 1$, $R_2 = 2$, $\Omega_1 = 1$ and $\gamma \geqslant 0$.

A brief description of the efforts due to changes of each parameter is given below.

Changes in u_1

For $u_1 < 0$ a strong departure of the solution is obtained from that for the case $u_1 = 0$.

The velocity profile becomes steeper and steeper at the inner wall as u_1 grows. The fluid particles are pushed towards the inner wall and v can overshoot values at the boundaries (Figure 3(a)). This never happens in the case without suction or injection (see Remark 2, in Section 4).

A similar effect is evidenced for $u_1 > 0$. The fluid particles are driven to the outer wall, but it turns out that there is no solution if u_1 exceeds some critical value (Figure 3(b)).

The shear stress (and also the torque and drag) at both walls is reduced with increasing (see Table I).

Changes in ν

As already seen in Section 5, a decrease in the kinematic viscosity ν causes the same effect as an increase in the modulus of the suction (or injection) velocity, i.e. a steepening of the velocity slope and a reduction of the shear stress at both walls (Figure 4).

Changes in λ_1 and λ_2

In Section 4 we saw that without a radial velocity applied to the boundary the solution obtained is identical for all values of λ_1 and λ_2. Even a small suction or injection velocity is insufficient to modify considerably the velocity field, but it activates a noticeable change in shear and normal stresses with λ_1 and λ_2. The shear stress decreases with increasing λ_1 and with decreasing λ_2 over the whole domain, as Table I shows. This behaviour is qualitatively similar in both cases of suction and injection.

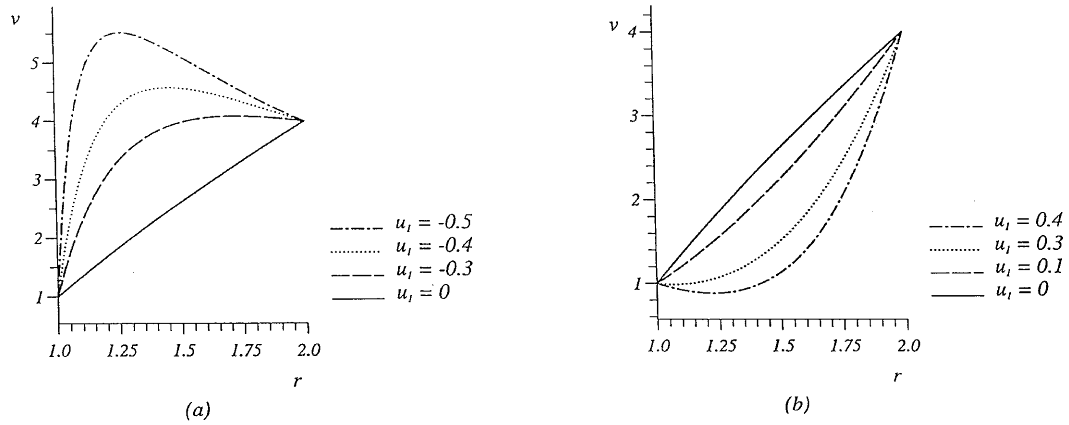

Figure 3. Velocity profiles for $\lambda_1 = 0{\cdot}4$, $\lambda_2 = 0$, $v = 0{\cdot}1$, $\Omega_2 = 2$ in case of (a) suction and (b) injection

Changes in Ω_2

The changing of the right boundary condition produces a variation in the whole flow and, as a consequence, in the shear stress all over. The latter becomes larger for larger Ω_2, as expected. The same phenomenon is shown in the suction and injection cases (Figure 5).

7. CONCLUSIONS

We studied the steady flow between two concentric rotating cylinders. Their walls are porous and a suction or injection velocity is applied at one of them. The fluid considered is the Oldroyd-B model. Apart from some special cases where an analytical solution is available, in the general case a numerical technique is used, which has revealed to be promising for such flow problems. The results presented show how the solution varies along the many flow and fluid parameters and exhibits a strong dependence on the radial velocity.

Table I. Shear stress at two walls for different values of parameters ($\Omega_2 = 2$)

λ_1	λ_2	v	u_1	$S_{r\theta}\|_{r=1}$	$S_{r\theta}\|_{r=\beta}$
0	0	0·5	−0·3	1·990	−0·028
0·1	0	0·5	−0·3	1·946	−0·039
0·4	0	0·5	−0·3	1·831	−0·067
0·8	0	0·5	−0·3	1·707	−0·098
0·8	0·2	0·5	−0·3	1·779	−0·080
0·8	0·4	0·5	−0·3	1·850	−0·063
0·8	0·6	0·5	−0·3	1·920	−0·045
0·4	0	1	−0·3	3·138	0·260
0·4	0	1	0·3	1·914	1·003
0·4	0	0·1	−0·3	1·033	−0·267
0·4	0	0·1	−0·1	0·480	−0·055
0·4	0	0·1	0	0·267	0·067
0·4	0	0·1	0·1	0·084	0·196
0·4	0	0·1	0·3	−0·164	0·484

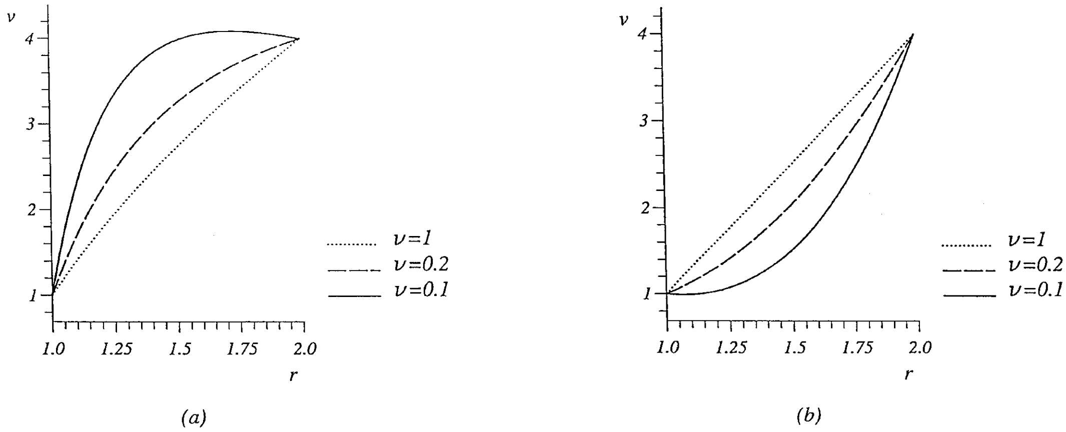

Figure 4. Velocity profiles for $\lambda_1 = 0{\cdot}4$, $\lambda_2 = 0$, $\Omega_2 = 2$, (a) $u_1 = -0{\cdot}3$, (b) $u_1 = 0{\cdot}3$

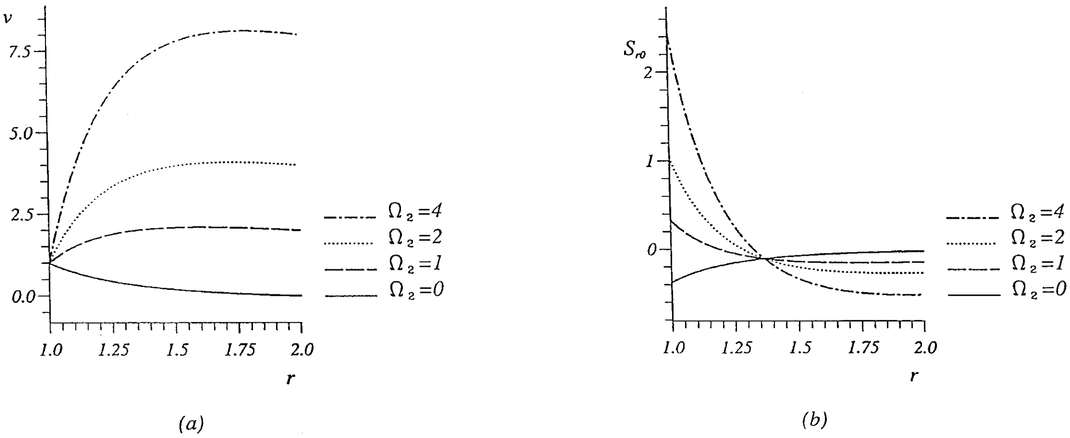

Figure 5. (a) Velocity profiles and (b) shear stress for $\lambda_1 = 0{\cdot}4$, $\lambda_2 = 0$, $\nu = 0{\cdot}1$, $u_1 = -0{\cdot}3$

REFERENCES

1. M. Couette, ‘Étude sur le frottment des liquides’, *Ann. Chim. Phys.*, **21**, (1890).
2. G. I. Taylor, ‘Stability of a viscous liquid contained between two rotating cylinders’, *Philos. Trans. R. Soc. Lond. A*, **223**, (1923).
3. P. Chossat and G. Iooss, *The Couette-Taylor Problem*, Springer-Verlag (1994).
4. G. K. Batchelor, *An Introduction to Fluid Dynamics*, Cambridge University Press, Cambridge, 1967.
5. H. Schlichting, *Boundary-Layer Theory*, McGraw-Hill, New York, 1968.
6. J. G. Oldroyd, ‘On the formulation of rheological equations of state’, *Proc. R. Soc. A*, **200**, 523–541, (1950).
7. J. G. Oldroyd, ‘The motion of an elastico-viscous liquid contained between coaxial cylinders. I’, *Q. J. Mech. Appl. Math.*, **4**, 271–282, (1951).
8. R. B. Bird, R. C. Armstrong and O. Hassager, *Dynamics of Polymeric liquids*, Vol. 1, *Fluid Mechanics*, Wiley, New York, 1987.
9. Y. Fukumoto, ‘General unsteady circulatory flow outside a porous circular cylinder with suction or injection’, *J. Phys. Soc. Jpn.*, **59**, 918–926, (1990).
10. J. E. Daskalakis, ‘Mixed free and forced convection in the incompressible boundary layer along a rotating vertical cylinder with fluid injection’, *Int. J. Energy Res.*, **17**, 689–695, (1993).
11. M. J. Casarella and P. A. Laura, ‘Drag on an oscillating rod with longitudinal and torsional motion’, *J. Hydronaut.*, **3**, 180–183, (1969).

12. F. U. Unikeev, 'Plastic behavior of superplastic material moving between two coaxial rotating cylinders', *Int. J. Mech. Sci*., **35**, 81–88, (1993).
13. N. Prasanth and U. Y. Shenoy, 'Poiseuille flow of a power-law fluid between coaxial cylinders', *J. Appl. Polym. Sci*., **46**, 1189–1194, (1992).
14. K. R. Rajagopal, 'Longitudinal and torsional oscillations of a rod in a non-newtonian fluid', *Acta Mech*., **49**, 281–285, (1983).
15. K. R. Rajagopal and P. N. Kaloni, 'Some remarks on boundary conditions for flows of fluids of the differential type', *Continuum Mechanics and Its Applications*, Hemisphere, Washington, DC, 1989.
16. G. Hamel, 'Spiralförmige Bewegungen zäher Flüssigkeiten', *Jber. Dtsch. MatVer*., **25**, 34–60, (1917).
17. J. H. Preston, 'The steady circulatory flow about a circular cylinder with uniformly distributed suction at the surface', *Aeronaut. Q*., **1**, 319–338, (1950).
18. B. Thwaites, 'Note on the circulatory flow about a circular cylinder through which the normal velocity is large', *Q. J. Mech*., **3**, 74–79, (1950).
19. C. E. Roberts Jr., *Ordinary Differential Equations: a Computational Approach*, Prentice-Hall, Englewood Cliffs, NJ, 1979.
20. C. de Boor, *A Practical Guide to Splines*, Springer-Verlag, 1978.
21. C. de Boor and B. Swartz, 'Collocation at Gaussian points', *SIAM J. Numer. Anal*., **10**, 582–606 (1973).
22. W. C. Rheinboldt, *Numerical Analysis of Parametrized Nonlinear Equations*, Wiley, New York, 1986.